

A COUNTEREXAMPLE ON SPECTRA OF ZERO PATTERNS

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ABSTRACT. An $n \times n$ zero pattern S , which is a matrix with entries $*$ and 0 , is called *spectrally arbitrary* with respect to a field \mathbb{F} if any monic polynomial f of degree n can be realized as the characteristic polynomial of a matrix obtained from S by replacing the $*$'s with non-zero elements of \mathbb{F} . We construct an $n \times n$ zero pattern that is spectrally arbitrary with respect to \mathbb{C} and has $2n - 1$ nonzero entries.

One of the intriguing questions in the theory of incomplete matrices is as follows. What is the smallest number k such that there exists an $n \times n$ zero pattern with k nonzero entries which is spectrally arbitrary over \mathbb{R} ? It was proved in [4] that k has to be at least $2n - 1$ in the above question, and we note that the proof works even if \mathbb{R} gets replaced by any other infinite field. On the other hand, many spectrally arbitrary $n \times n$ zero patterns with $2n$ non-zero entries are known (see [5]), so the optimal value of k is at most $2n$. The statement that this optimal value is in fact $2n$ has become known¹ as the *2n conjecture* (see [3]), and it attracts a significant amount of attention in the contemporary linear algebra community ([1, 2, 4, 6]).

This paper is a piece of evidence against this conjecture. Although we were not able to disprove it so far (and we explain why in the end of the paper), we present a counterexample to its complex analogue. Also, our result allows us to answer several questions asked by McDonald and Yielding in [6].

Now we proceed with a counterexample. Throughout our paper, we denote by x_1, \dots, x_8 a family of variables that are allowed to take any complex value except zero. We consider the matrix $X(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ defined as

$$\begin{pmatrix} x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_7 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x_8 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x_6 & 0 & 0 & x_5 & 0 & x_4 & 0 & 0 \end{pmatrix},$$

and we denote its zero pattern by S . It is easy to check that the matrix $X(1, -1, 1, 1, -1, 1, -2, 1)$ is nilpotent, so we can realize t^8 as the characteristic polynomial of a matrix with zero pattern S . The polynomial $(x - 1)^8$ is a bit harder to realize: One needs to take $x_1 = 1737/848$, $x_2 = 5047/848$, $x_3 = -4452/193$, $x_4 = 35/4$, $x_5 = 2/7$, $x_6 = 25/2$, $x_7 = 1007374319/138787072$, $x_8 = -1325/7$,

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¹Strictly speaking, the *2n conjecture* is the same question but asked for the sign patterns instead of zero patterns.

turn on the computer, and check that the matrix X defined this way has a desired characteristic polynomial.

How to describe all the characteristic polynomials realized by S ? First of all, we note that any matrix with pattern S can be reduced to the form X by conjugating it with a diagonal matrix. Since a pair of similar matrices have the same characteristic polynomial, we can restrict our attention to the matrices of the form X . We use the computer again and compute $\varphi = \det(tI - X)$, which is a monic polynomial $t^8 + \varphi_7 t^7 + \dots + \varphi_0$ with coefficients φ_i in $\mathbb{C}[x_1, \dots, x_8]$. As we look into the result of the computation, we note that φ_4 is a multiple of φ_7 . In particular, φ_7 cannot be zero unless φ_4 is zero, so S is not a spectrally arbitrary pattern.

Despite this fact, a lot of polynomials can be realized by the pattern S . To see this, we consider new variables τ_0, \dots, τ_7 and find the simultaneous solution of the equations $\varphi_i = \tau_i$ for x_1, \dots, x_8 . This may sound as a hard task, but the computer comes up with the solution immediately, — this becomes possible thanks to a carefully selected pattern S . The resulting values of the x_i 's are all rational functions in the τ_i 's, so the polynomial $t^8 + \tau_7 t^7 + \dots + \tau_0$ can be realized by S unless one of these rational functions has a vanishing numerator or denominator. Taking the LCM of all these numerators and denominators, we get a polynomial $\pi(\tau_0, \dots, \tau_7)$ which can vanish only if $t^8 + \tau_7 t^7 + \dots + \tau_0$ cannot be realized by S .

Now we have learned everything we need about the possible characteristic polynomials of X , and we turn our attention to its *spectra*. For any family σ of eight complex numbers, we define s_i to be $(-1)^i$ times the i th elementary symmetric polynomial of σ and define $\psi(\sigma) = \pi(s_8, s_7, \dots, s_1)$. One can determine the total degree² of ψ (which equals 94), and Vieta's formulae show that σ is the spectrum of a matrix of the form X whenever ψ does not vanish.

Since ψ is a polynomial in eight variables and has total degree 94, every set of $94 + 8 = 102$ distinct complex numbers has a subset σ of eight elements which satisfies $\psi(\sigma) \neq 0$. Therefore, if a family U of 708 complex numbers contains at least 102 distinct elements, then it has a subset realizable as the spectrum of a matrix of the form X . Otherwise, we use the pigeonhole principle and conclude that some number c repeats in U at least eight times. As explained above, $(x - c)^8$ is realizable as the characteristic polynomial of X , so U does anyway contain a subfamily V realizable as the spectrum of a matrix M with pattern S .

Now we define D_{2m} to be the block-diagonal matrix consisting of the m blocks equal to the 2×2 matrices of all $*$'s. It is easy to see that D_{2m} is spectrally arbitrary, so we can find a matrix M' with pattern D_{700} and spectrum $U \setminus V$. Now we see that the matrix $\text{diag}(M, M')$ has spectrum U and pattern $\text{diag}(S, D_{700})$. Therefore, $\text{diag}(S, D_{700})$ is a 708×708 zero pattern which has 1415 nonzero elements and is spectrally arbitrary with respect to \mathbb{C} .

As said above, our result answers two questions asked in [6]. First, we have constructed an $n \times n$ zero pattern which has $2n - 1$ nonzero entries and is spectrally arbitrary with respect to \mathbb{C} . Secondly, we get an example of zero patterns A, B such that $\text{diag}(A, B)$ is spectrally arbitrary but A is not. We note in passing that an argument similar to the proof of item (3) of Theorem 11 in [7] would allow us to get a refined version of this result. Namely, we would be able to show that $\text{diag}(P, P)$ can be a spectrally arbitrary pattern even if P is not spectrally arbitrary.

²Of course, we do not need to calculate ψ to do this. Instead, we compute the maximal weight of the monomials of φ with respect to the function $(\tau_0, \dots, \tau_7) \rightarrow (8, 7, \dots, 1)$.

We conclude our paper with several thoughts on the real version of the $2n$ conjecture. My attempts to disprove it were not successful, and the main obstacle was the fact that the set of real irreducible polynomials is much richer than its complex counterpart. A brief examination of our counterexample allows one to prove the following sufficient condition for S to be a diagonal block of a block-diagonal pattern spectrally arbitrary over \mathbb{C} . Namely, this happens if the set of all characteristic polynomials allowed by S contains t^n , $(t-1)^n$, and a generic monic polynomial of degree n . Unfortunately, this condition is not sufficient for real patterns, and the reason lies in the existence of degree-two irreducible polynomials over \mathbb{R} . The set of such polynomials remains positive-dimensional even if the spectra are considered up to scaling, which makes it hard to produce families of matrices that depend on n parameters only and allow all such polynomials. For instance, there exist (finitely many) values of $a \in (-2, 2)$ for which $(t^2 + at + 1)^4$ cannot be the spectrum of the matrix X as above. In particular, this happens if

$$a = 0 \text{ or } a = \sqrt{\frac{\sqrt{15} - 3}{3}}$$

and makes it impossible for S to be a diagonal block of a block-diagonal pattern spectrally arbitrary over \mathbb{R} . However, we believe that a counterexample for the real $2n$ conjecture can be found with a more extensive search.

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